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# Low energy effective action of domain-wall fermion and the Ginsparg-Wilson relation

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We derive the effective action of the light fermion field of the domain-wall fermion, which is referred as  $q(x)$  and  $\bar{q}(x)$  by Furman and Shamir. The inverse of the effective Dirac operator turns out to be identical to the inverse of the truncated overlap Dirac operator except a local contact term, which would give the chiral symmetry breaking in the Ginsparg-Wilson relation. We argue that there are direct relations between the low energy observables of the domain-wall QCD and observables of the Ginsparg-Wilson fermion described by the (truncated) overlap Dirac operator.

## 1. Introduction

Recently, our understanding of chiral symmetry on the lattice has substantially improved. Lattice Dirac operators have been obtained [1,2],

$$S_F = a^4 \sum_x \bar{\psi}(x) D \psi(x), \quad (1)$$

which are gauge covariant, define local actions [3] and satisfy the Ginsparg-Wilson relation [4],

$$\gamma_5 D + D \gamma_5 = a D R \gamma_5 D. \quad (2)$$

The Ginsparg-Wilson relation implies the exact chiral symmetry of the action under the transformation [5]

$$\delta \psi(x) = \gamma_5 (1 - a R D) \psi(x), \quad \delta \bar{\psi}(x) = \bar{\psi}(x) \gamma_5. \quad (3)$$

The goal of this paper is to understand the chiral property of the light fermion of the domain-wall QCD from the point of view of the exact chiral symmetry based on the Ginsparg-Wilson relation. The domain-wall fermion [6] is the basis of Neuberger's overlap Dirac operator [1]. In a simplified formulation [7,8], the domain-wall fermion consists of  $N$ -flavor Wilson fermions with

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a certain flavor-mixing mass matrix. Due to its structure of the chiral hopping and the boundary condition in the flavor space, a single light Dirac fermion can emerge in the spectrum. This light fermion can be probed suitably by the field variables at the boundary in the flavor space, which are referred as  $q(x)$  and  $\bar{q}(x)$  by Furman and Shamir:

$$\begin{aligned} q(x) &= \psi_{1L}(x) + \psi_{NR}(x), \\ \bar{q}(x) &= \bar{\psi}_{1L}(x) + \bar{\psi}_{NR}(x). \end{aligned} \quad (4)$$

It has been argued that the chiral symmetry of the light fermion is preserved up to corrections suppressed exponentially in the number of flavors [8]. The chiral transformation of the light fermion field in this context is defined as

$$\delta q(x) = \gamma_5 q(x), \quad \delta \bar{q}(x) = \bar{q}(x) \gamma_5. \quad (5)$$

It has been known that in the subtraction scheme proposed by Vranas [9], the partition function of the domain-wall fermion reduces to a single determinant of the truncated overlap Dirac operator [10]:

$$D_N = \frac{1}{2a} \left( 1 + \gamma_5 \tanh \frac{N}{2} a_5 \tilde{H} \right), \quad (6)$$

where  $\tilde{H}$  is defined through the transfer matrix of the five-dimensional Wilson fermion with a nega-

tive mass. (In this derivation, the positivity of

$$B = 1 + a_5 \left( -\frac{a}{2} \nabla_\mu \nabla_\mu^* - \frac{m_0}{a} \right) \quad (7)$$

is required for the transfer matrix to be defined consistently. It is assured when  $0 < \frac{a_5}{a} m_0 < 1$ . It is also assumed that  $N$  is even. )

In this paper, we will show that there is a simple correspondence between the light fermion field of the domain-wall fermion and the Dirac field described by the (truncated) overlap Dirac operator:

$$q(x), \bar{q}(x) \iff \left( 1 - \frac{a}{2} R D_N \right) \psi(x), \bar{\psi}(x). \quad (8)$$

We will then discuss the relation between the low energy observables of the domain-wall fermion and those of the Ginsparg-Wilson fermion described by the (truncated) overlap Dirac operator. [11]

## 2. Effective action of the light fermion

For this purpose, we first derive the low energy effective action of the light fermion field by integrating out  $N-1$  heavy flavors of the domain-wall fermion:

$$S_N^{\text{eff}} = a^4 \sum_x \bar{q}(x) D_N^{\text{eff}} q(x). \quad (9)$$

Since the inverse of this effective Dirac operator should give the propagator of  $q(x)$  and  $\bar{q}(x)$ , this can be achieved by calculating the propagator. The result turns out that [11]

$$\begin{aligned} D_N^{\text{eff}}(x, y) &= \frac{1}{a^4} \langle q(x) \bar{q}(y) \rangle^{-1} \\ &= \frac{1}{a_5} \frac{1 + \gamma_5 \tanh \frac{N}{2} a_5 \tilde{H}}{1 - \gamma_5 \tanh \frac{N}{2} a_5 \tilde{H}}. \end{aligned} \quad (10)$$

This result implies that the inverse of the effective Dirac operator gives the inverse of the truncated overlap Dirac operator up to a local contact term:

$$\frac{a}{a_5} D_N^{\text{eff}-1} + a \delta(x, y) = D_N^{-1}. \quad (11)$$

This contact term just takes account of the chiral symmetry breaking in the Ginsparg-Wilson relation, which holds true for the overlap Dirac operator in the limit of the infinite flavors.

## 3. Relation between $q(x)$ $\bar{q}(x)$ and $\psi(x)$ $\bar{\psi}(x)$

After integrating out  $N-1$  heavy flavors of the domain-wall fermion and also corresponding  $N-1$  flavors of the Pauli-Villars boson, the partition function of the domain-wall fermion assumes the following expression in functional integral:

$$\bar{Z}_{\text{DW}} = \int [dq d\bar{q}] [dQ d\bar{Q}] e^{-\bar{S}_N^{\text{eff}}[q, \bar{q}, Q, \bar{Q}]}, \quad (12)$$

where we have denoted the field variables of the Pauli-Villars fields at the boundary in the flavor space by  $Q(x)$  and  $\bar{Q}(x)$ . The total effective action is given by

$$\begin{aligned} \bar{S}_N^{\text{eff}}[q, Q] &= a^4 \sum_x \bar{q}(x) D_N^{\text{eff}} q(x) \\ &+ a^4 \sum_x \bar{Q}(x) \left\{ D_N^{\text{eff}} + \frac{1}{a_5} \right\} Q(x). \end{aligned} \quad (13)$$

Note that  $Q(x)$  and  $\bar{Q}(x)$  acquires the unit mass of order  $1/a_5$  because of the anti-periodic boundary condition in the flavor space. Since Eq. (11) implies the following identity

$$\frac{a_5 D_N^{\text{eff}}}{1 + a_5 D_N^{\text{eff}}} = a D_N, \quad (14)$$

we see that the partition function can be rewritten into

$$\bar{Z}_{\text{DW}} = \int [d\psi d\bar{\psi}] e^{-a^4 \sum_x \bar{\psi}(x) D_N \psi(x)} \quad (15)$$

through the change of the field variables along the relation

$$\begin{aligned} q(x) &= Z \frac{1}{1 + a_5 D_N^{\text{eff}}} \psi(x) \\ &= Z \left( 1 - \frac{a}{2} R D_N \right) \psi(x), \\ \bar{q}(x) &= \bar{\psi}(x), \quad Z = \frac{a_5}{a}, R = 2. \end{aligned} \quad (16)$$

The Jacobian associated with the change of the field variables just compensates the determinant resulting from the integration of  $Q(x)$   $\bar{Q}(x)$ . Thus we reproduce the result of [10].

An immediate consequence of Eq. (16) is that any local observables of the domain-wall fermion written in terms of only  $q(x)$  and  $\bar{q}(x)$  can be related to local observables written in terms of

$\psi(x)$ ,  $\bar{\psi}(x)$  and  $D_N$ , as long as  $D_N$  assumes a local Dirac operator:

$$\mathcal{O}_{\text{DW}}^{(N)}[q, \bar{q}] = \mathcal{O}_{\text{GW}}^{(N)}[\psi, \bar{\psi}; D_N]. \quad (17)$$

Then we can relate the chiral properties of these observables to the (would-be) exact chiral symmetry based on the Ginsparg-Wilson relation. A typical example is given by the chiral multiplet of scalar and pseudo scalar bilinear operators: from Eq. (16), we obtain a relation among them,

$$\begin{aligned} \bar{q}(x)q(x) &= Z \bar{\psi}(x) \left(1 - \frac{a}{2} R D_N\right) \psi(x), \\ \bar{q}(x)\gamma_5 q(x) &= Z \bar{\psi}(x)\gamma_5 \left(1 - \frac{a}{2} R D_N\right) \psi(x). \end{aligned}$$

Note that operators in the r.h.s. consist the exact chiral multiplet of the chiral transformation Eq. (3) in the limit of infinite flavors  $N \rightarrow \infty$ , as discussed by Niedermayer [12].

#### 4. Axial-vector current and chiral anomaly

Let  $\bar{A}_{\mu\text{DW}}^a(x)$  and  $\tilde{A}_{\mu\text{GW}}^a(x)$  denote the axial-vector currents of the domain-wall fermion and of the Ginsparg-Wilson fermion described by Neuberger's Dirac operator with  $\tilde{H}$ , respectively. An important aspect of  $\bar{A}_{\mu\text{DW}}^a(x)$  we should note is that it is written not only in terms of the light fermion field, but also in terms of all heavy flavors including the Pauli-Villars fields. Using Eq. (11), it is possible to derive the following identities which relate them:

$$\lim_{N \rightarrow \infty} \langle \bar{A}_{\mu\text{DW}}^a(x) \rangle = \langle \tilde{A}_{\mu\text{GW}}^a(x) \rangle, \quad (18)$$

$$\begin{aligned} \lim_{N \rightarrow \infty} \langle q(y) \bar{A}_{\mu\text{DW}}^a(x) \bar{q}(z) \rangle_c \\ = Z \langle \psi(y) \tilde{A}_{\mu\text{GW}}^a(x) \bar{\psi}(z) \rangle_c. \end{aligned} \quad (19)$$

On the other hand, the term which breaks chiral symmetry in the axial Ward-Takahashi identity is given in terms of the heavy flavors:

$$\partial_\mu^* \langle \bar{A}_{\mu\text{DW}}^a(x) \rangle = \frac{2}{a_5} \langle \bar{\psi}'_{\frac{N}{2}}(x) \gamma_5 \psi'_{\frac{N}{2}}(x) \rangle + \dots \quad (20)$$

The r.h.s. times  $a^4$  can be evaluated as

$$\begin{aligned} (\text{r.h.s.}) &= -a \text{tr} \gamma_5 R D_N(x, x) \\ &\quad - \text{tr} \left( D_N^{-1} \gamma_5 \frac{1}{a} \frac{1}{\cosh^2 \frac{N}{2} a_5 \tilde{H}} \right) (x, x). \end{aligned} \quad (21)$$

Then we can see that the chiral symmetry breaking reduces to the chiral anomaly of the Ginsparg-Wilson fermion in the limit  $N \rightarrow \infty$ ,

$$-a \text{tr} \gamma_5 R \tilde{D}(x, x). \quad (22)$$

#### 5. Conclusion

In conclusion, the light fermion field introduced by Furman and Shamir has a rather direct and simple relation to the Ginsparg-Wilson fermion field described by the (truncated) overlap Dirac operator. The chiral properties of the low energy observables of the domain-wall fermion can be interpreted in terms of the exact chiral symmetry based on the Ginsparg-Wilson relation.

This direct relation can be extended to the case of chiral fermions: coupled to an interpolating five-dimensional gauge field, the complex phase of the partition function of the domain-wall fermion can be regarded as a lattice implementation of the eta-invariant and it has a direct relation to the effective action of the chiral Ginsparg-Wilson fermion, which is defined through the chirality  $\hat{\gamma}_5 = \gamma_5(1 - aRD)$ . See [13] for detail.

#### REFERENCES

1. H. Neuberger, Phys. Lett. B417 (1998) 141; Phys. Lett. B427 (1998) 353.
2. P. Hasenfratz, Nucl. Phys. B Proc. Suppl. 63 (1998) 53.
3. P. Hernández, K. Jansen and M. Lüscher, Nucl. Phys. B552 (1999) 363.
4. P. H. Ginsparg and K. G. Wilson, Phys. Rev. D25 (1982) 2649.
5. M. Lüscher, Phys. Lett. B428 (1998) 342.
6. D. B. Kaplan, Phys. Lett. B288 (1992) 342.
7. Y. Shamir, Nucl. Phys. B406 (1993) 90.
8. V. Furman and Y. Shamir, Nucl. Phys. B439 (1995) 54.
9. P. Vranas, Phys. Rev. D57 (1998) 1415.
10. H. Neuberger, Phys. Rev. D57 (1998) 5417.
11. Y. Kikukawa and T. Noguchi, hep-lat/9902022.
12. F. Niedermayer, Nucl. Phys. B Proc. Suppl. 73 (1999) 105.
13. T. Aoyama and Y. Kikukawa, hep-lat/9905003.